PLANE-PARALLEL FLOW OF A COMPRESSIBLE FLUID IN THE WAKE BEHIND A BODY

(PLOSKO-PARALLEL'NOE TECHENIE SZHIMAEMOI ZHIDKOSTI V SLEDE ZA TELOM)

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Let a plane-parallel stream of a viscous compressible fluid with velocity u_∞ = const flow toward a fixed body which is symmetric with respect to the flow direction. At large distances from the body in the wake the pressure is approximately constant in transverse sections of the wake, the transverse velocity is small in comparison with the longitudinal velocity and the rate of change of the longitudinal velocity along the axis of the wake is small in comparison with its rate of change in the transverse section. Therefore, in an unbounded fluid, the pressure gradient along the axis of the wake is negligibly small. Then we have the following basic equations:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \text{ (equation of motion)} \tag{1}$$

$$\rho u \frac{\partial (C_p t)}{\partial x} + \rho v \frac{\partial (C_p t)}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial t}{\partial y}\right) + \frac{\mu}{J}\left(\frac{\partial u}{\partial y}\right)^2 \text{ (energy equation)} \tag{2}$$

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \text{ (continuity equation)} \tag{3}$$

$$\rho t = \rho_\infty t_\infty \text{ (equation of state)} \tag{4}$$

Here the coordinate x lies along the axis of symmetry, u, v are the components of the fluid velocity along the coordinate axes, ρ is the fluid density, μ the viscosity, t the temperature, C_p the specific heat at constant pressure, k the coefficient of thermal conductivity and J the mechanical equivalent of heat. The subscript ∞ denotes parameters in the undisturbed flow. We assume that

$$C_p = \text{const}, \qquad Pr = \frac{C_p \mu}{k} = 1, \qquad \frac{\mu}{\mu_\infty} = \left(\frac{t}{t_\infty}\right)^m \qquad (m = \text{const}) \tag{5}$$

In this case the energy equation can be integrated (Crocco):

$$t = A + Bu - \frac{u^2}{2C_pJ} \tag{6}$$

where A and B are undetermined constants. We introduce the stream function by the formulas

$$u = \frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial y}, \quad v = -\frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial x}$$

and change variables from x, y to x, ψ. We have

$$\left(\frac{\partial}{\partial y}\right)_x = \frac{\rho u}{\rho_\infty}\left(\frac{\partial}{\partial\psi}\right)_x, \qquad \left(\frac{\partial}{\partial x}\right)_y = -\frac{\rho v}{\rho_\infty}\left(\frac{\partial}{\partial\psi}\right)_x + \left(\frac{\partial}{\partial x}\right)_\psi$$

Then Equation (1) assumes the form

$$\rho_\infty\frac{\partial u}{\partial x} = \frac{\partial}{\partial\psi}\left(\mu u\frac{\rho}{\rho_\infty}\frac{\partial u}{\partial\psi}\right) \tag{7}$$

At large distances from the body in the wake $u \approx u_\infty + u_1$, $v \approx v_1$, where u_1, v_1 are small. Confining ourselves to the main terms, we have, instead of (7)

$$\rho_\infty\frac{\partial u_1}{\partial x} = u_\infty\frac{\partial}{\partial\psi}\left(\frac{\mu\rho}{\rho_\infty}\frac{\partial u_1}{\partial\psi}\right) \tag{8}$$

We introduce the dimensionless quantities by the formulas

$$U_1 = \frac{u_1}{u_\infty}, \qquad T = \frac{t}{t_\infty}, \qquad X = \frac{x}{L}, \qquad \Psi = \frac{\psi}{\sqrt{u_\infty\nu_\infty L}} \qquad \left(\nu = \frac{\mu}{\rho}\right)$$

Here L is a characteristic dimension. By virtue of (4) and (5)

$$\frac{\mu\rho}{\rho_\infty} = \mu_\infty T^{m-1}$$

Then (8) assumes the form

$$\frac{\partial U_1}{\partial X} = \frac{\partial}{\partial\Psi}\left(T^{m-1}\frac{\partial U_1}{\partial\Psi}\right) \tag{9}$$

This equation admits an analytical solution if $m = 1$. In this case

$$\frac{\partial U_1}{\partial X} = \frac{\partial^2 U_1}{\partial\Psi^2} \tag{10}$$

The boundary conditions for Equations (1) to (4) are

$$v = 0,\ \frac{\partial u}{\partial y} = 0 \quad \text{for } y = 0, \qquad u \to u_\infty,\ v \to 0,\ t \to t_\infty \quad \text{for } y \to \pm\infty \tag{11}$$

From this

$$U_1 = 0 \quad \text{for } \Psi = \infty, \qquad \frac{\partial U_1}{\partial \Psi} = 0 \quad \text{for } \Psi = 0 \tag{12}$$

if the axis of symmetry is taken as the streamline $\psi = 0$.

We enclose the body in some control volume AA_1B_1B, so chosen that AB and A_1B_1 lie at a large distance h from the body in the undisturbed flow and are parallel to the undisturbed flow velocity, AA_1 lies ahead of the body in the undisturbed flow and perpendicular to its velocity and BB_1 lies behind the body and parallel to AA_1.

The total momentum flow across the control surface equals

$$\int_{-h}^{h} \rho u u_1 dy$$

If D is the drag per unit thickness of the obstacle, then by the momentum theorem

$$D = \int_{-h}^{h} \rho u u_1 dy, \qquad \text{or} \qquad D = \int_{-\infty}^{\infty} \rho u u_1\, dy \sim \int_{-\infty}^{\infty} U_1 d\Psi \tag{13}$$

Replacing $\pm h$ by $\pm\infty$ is permissible since $u_1 = 0$ for $|y| \geqslant h$.

Let

$$\zeta = \Psi / \sqrt{X}, \qquad U_1 = CX^q g(\zeta) \tag{14}$$

where C and q are constants. Then

$$D \sim \int_{-\infty}^{\infty} X^q \sqrt{X} g(\zeta)\, d\zeta$$

But D is a constant quantity, hence the integral must be independent of X. Consequently, $q = -1/2$. Then by (14), we have

$$U_1 = CX^{-1/2} g(\zeta) \tag{15}$$

and instead of (10) we have

$$g'' + \frac{1}{2}\zeta g' + \frac{1}{2} g = 0 \tag{16}$$

The boundary conditions are

$$g' = 0 \quad \text{for } \zeta = 0, \qquad g \to 0 \quad \text{for } \zeta \to \infty \tag{17}$$

Integrating (16) twice, and taking into account the boundary conditions, we obtain

$$g = \exp\left(-\frac{1}{4}\zeta^2\right) \tag{18}$$

The constants A, B and C are determined from conditions at infinity (11), Equation (13) if D is known and from the theorem of energy change applied to the contour AA_1B_1B. The temperature t is determined from Formula (6), and the density ρ from Equation (4).

The analogous problem for incompressible fluids was solved by Tollmien.

BIBLIOGRAPHY

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